A Simple Taxonomy of Multilayer Networks

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***Abstract*—*one of the major goals of multilayer networks is providing proper and suitable representations of complex systems with many interdependent components, which, in turn, might interact through many different channels. However, the terminology referring to systems with multiple different relations has not yet reached a consensus. This work introduces a simple taxonomy of multilayer networks. Four different dimensions characterize the basic metrics of structural properties. Based on the taxonomy, an appropriate formal definition is presented.*

Keywords—*complex systems, computer networks, multilayer networks, taxonomy.*

I. Introduction

Many real-world complex systems (such as biological, social and engineering networks) can be naturally represented by graphs [1]. As a consequence, graph analysis as powerful mathematical tools for modelling pairwise relationships among sets of objects/entities has become crucial to understand the characteristics of these systems. However, graphs traditionally capture only a single form of relationships between objects [1], whereas complex systems usually rely on different forms of such relationships when a number of coexisting topologies interact and depend on each other. In turn, multilayer networks constitute the natural environment and explicitly define different categories of relationships: each connection (such as information channel, activity or category) is represented by a layer and the same object/entity might have connections of different kind on each layer [2]. Assuming that all layers are informative, they can provide complementary information. Thus, it can be expected that a proper combination of the information contained in the different graph layers can cover the most important characteristics of complex systems.

Nevertheless, it is important to note that the terminology referring to systems with multiple different relations has not yet reached a consensus – different papers from various areas represent similar terminologies to refer to different models, or distinct names for the same model.

To fill the gap, this work introduces: (1) a simple taxonomy of multilayer networks based on their structural properties, (2) an appropriate formal definition which is completely compatible with the proposed taxonomy.

The rest of this paper is structured as follows. Section 2 introduces the related work. Section 3 presents the taxonomy of multilayer networks. In turn, Section 4 focuses on the formal basic definition of multilayer networks. Finally, conclusion remarks are given in Section 5.

II. Related Work

As a result, interdisciplinary efforts of the last fifteen years with the aim of extracting the ultimate and optimal representation of complex systems and their underlying mechanisms have led to the birth of a movement in science, nowadays well-known as complex networks theory [3][4][5]. The main goals are[2]:

- the extraction of unifying principles that could encompass and describe (under some generic and universal rules) the structural accommodation;
- the modeling of the resulting emergent dynamics to explain what can be actually seen from the observation of such systems.

The traditional complex network approach is concentrated on cases when each system elementary unit (node or entity) is charted into a network node (graph vertex), and each unit–to–unit interaction (channel) is represented as a static link (weighted graph edge) that encapsulates all connections between units [6][7][8].

However, it is easy to realize that the assumption of encapsulation of different types of communication into a single link is almost always a gross oversimplification and, as a consequence, it can lead to incorrect descriptions of some phenomena that are taking place on real-world networks.

As mention above, multilayer networks [8][9][2]explicitly incorporate multiple channels of connectivity and constitute the natural environment to describe systems interconnected through different types of connections: each channel (relationship, activity, category, etc.) is represented by a layer and the same node or entity may have different kinds of interactions (different set of neighbours in each layer).

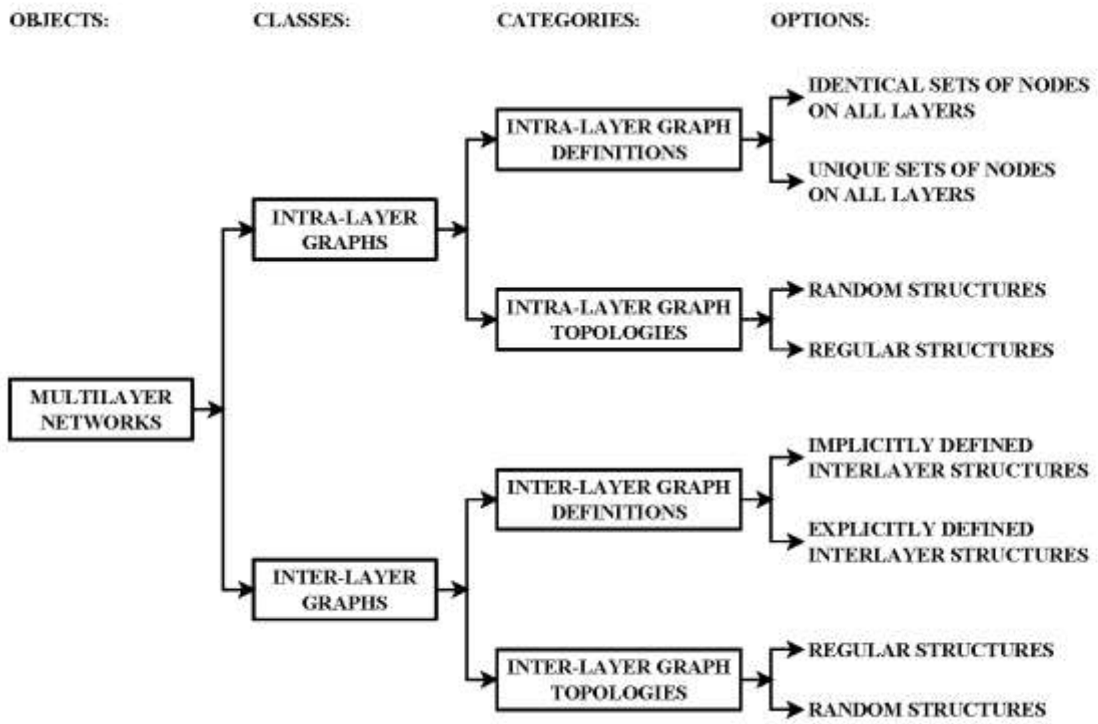

Fig. 1 A simple taxonomy of multilayer networks.

Recent surveys in the domain of multilayer networks provided by Kivela et al. [9] and Boccaletti et al. [2]give a comprehensive overview of the existing technical literature and summarize the properties of various multilayer structures.

III. TAXONOMY OF MULTILAYER NETWORKS

The proposed simple taxonomy of multilayer networks is shown in Fig. 1. All categories are decomposed into further elements that influence each other within or between categories. In general, this taxonomy strictly relies on the following notions:

- The definition of graphs as a collection of nodes (vertices) that can be connected to each other [1], i.e. the definition of basic metrics to characterize the structural properties of a graph as: (1) a set of nodes; and (2) the way in which these nodes interact (or topology).
- The definition of two main elements of multilayer networks as: (1) intra-layer graphs; and (2) inter-layer graphs [9].

As a consequence, two general classes are identified as follows: (1) intra-layer graphs, and (2) inter-layer graphs. In turn, each of the classes is divided into further categories. The class of intra-layer graphs consists of: (1) intra-layer graph definitions; and (2) intra-layer graph topologies. The class of inter-layer graphs consists of: (3) inter-layer graph definitions; and (4) inter-layer graph topologies. In the context of this work, each category represents two possible options as follows:

1. Intra-layer graph definitions:
   - Identical sets of nodes on all layers. This case represents a fixed set of nodes connected by different types of links (see Fig. 2). For example: social [1][9]and transport [9] networks.
   - Unique sets of nodes on all layers. This case represents different sets of nodes connected by different types of links (see Fig. 3 and Fig. 4). For example: communication (computer) networks [1].
2. Intra-layer graph topologies:
   - Random structures based on Erdö s-Rényi (ER) [1]and/or Barabási–Albert (BA) [1]networks. For example: social and biological (epidemic) networks.

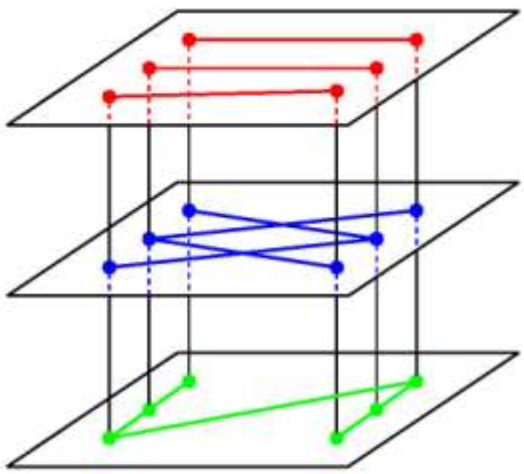

Fig. 2 A multilayer network with the identical sets of nodes on all layers.

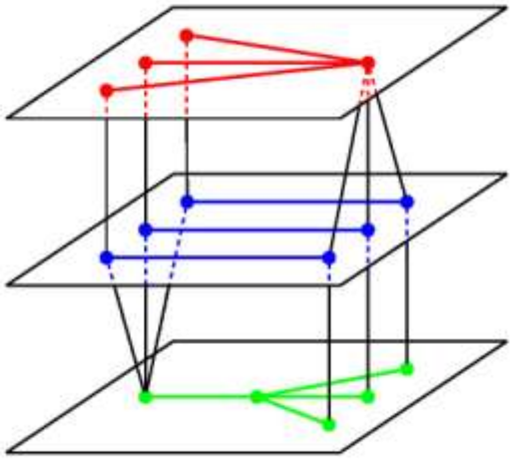

Fig. 3 A multilayer network with the unique sets of nodes on all layers(Example 1).

- Regularstructuresbased on topological models [10]. For example: all possible topologies of communication (computer) and transport networks.

3. Inter-layer graph definitions:
- Implicitly defined interlayer structures. This case represents the type of interlayer connections in which a given node is only connected to its counterpart nodes on the rest of layers (or one-to-one interlayer connections–see Fig. 2 and Fig. 4). In this case, inter-layer graphs are represented by biregular $(d = 1)$ balanced bipartite graphs and, as a consequence, they can be eliminated from formal definitions. For example: social, transport and biological networks.
- Explicitly defined interlayer structures. In this case, inter-layer graphs can represent the technological solutions (virtualization, replication, clustering, etc.– see Fig. 3) which were used to build the complex system [11]. For example: communication (computer) networks.

4. Inter-layer graph topologies:
- Random structures. The example is biological (epidemic) networks.
- Regular structures. The example is the topologies of communication (computer), social and transport networks.

IV. FORMAL DEFINITION

Based on the definitions of multilayer structures (intra-layer and inter-layer graphs), the taxonomy options which represent structural properties can be shown as a grid (see Fig. 5).

The most universal case – the intersection pointdenoted by combination of: (1) unique sets of nodes on all layers; and (2) explicitly defined interlayer structures(see Fig. 5)– represents the formal basic definition given by Boccaletti et al. [2]. In this case, multilayer networks can be defined as graphs:

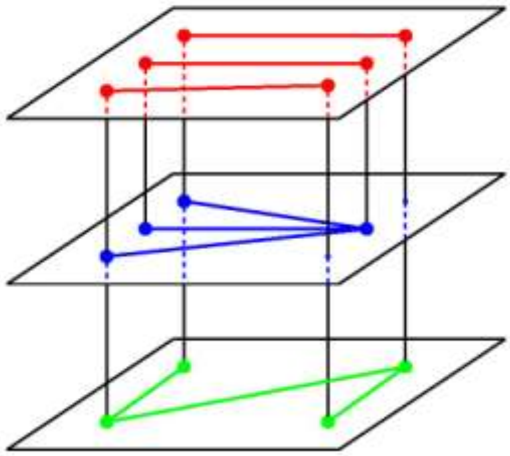

Fig. 4A multilayer network with the unique sets of nodes on all layers (Example 2).

$$M = (V, E)$$

where$M$ is a multilayer network or 3D graph; $V(M)$ is a finite, non-empty set of nodes; and $E(M)$ is a finite, non-empty set of node-to-node connections. In turn:

$$V(M) = \bigcup_{\alpha=1}^{N} V^{\alpha}$$

$$E(M) = \left(\bigcup_{\alpha=1}^{N} E^{\alpha}\right) \bigcup \left(\bigcup_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} E^{\alpha,\beta}\right)$$

where V^{α} is a finite, non-empty set of nodes on a given layer α; $E^{\alpha} \subseteq V^{\alpha} \times V^{\alpha}$ is a finite, non-empty set of intralayer connections on layer α; $E^{\alpha,\beta} \subseteq V^{\alpha} \times V^{\beta}$; $\alpha \neq \beta$ is a finite, non-empty set of interlayer connections between nodes on layer α and nodes on layer β; and N is the finite number of layers.

Based on the definitions of: (1) intra-layer graphs as $G^{\alpha} = (V^{\alpha}, E^{\alpha})$; and (2) inter-layer graphs as $G^{\alpha,\beta} = (V^{\alpha}, V^{\beta}, E^{\alpha,\beta})$; $\alpha \neq \beta$ multilayer networks can be represented as follows [11]:

$$M = \left(\bigcup_{\alpha=1}^{N} G^{\alpha}\right) \bigcup \left(\bigcup_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} G^{\alpha,\beta}\right)$$

This case covers(see Fig. 5):
- Heterogeneous networks [12][13];
- Interconnected networks[14][15];
- Interacting networks [16]
- Interdependent networks [17][18];
- Network of networks [19].

In turn, the other three intersection points can be described as special cases of the basic definition, i.e.:

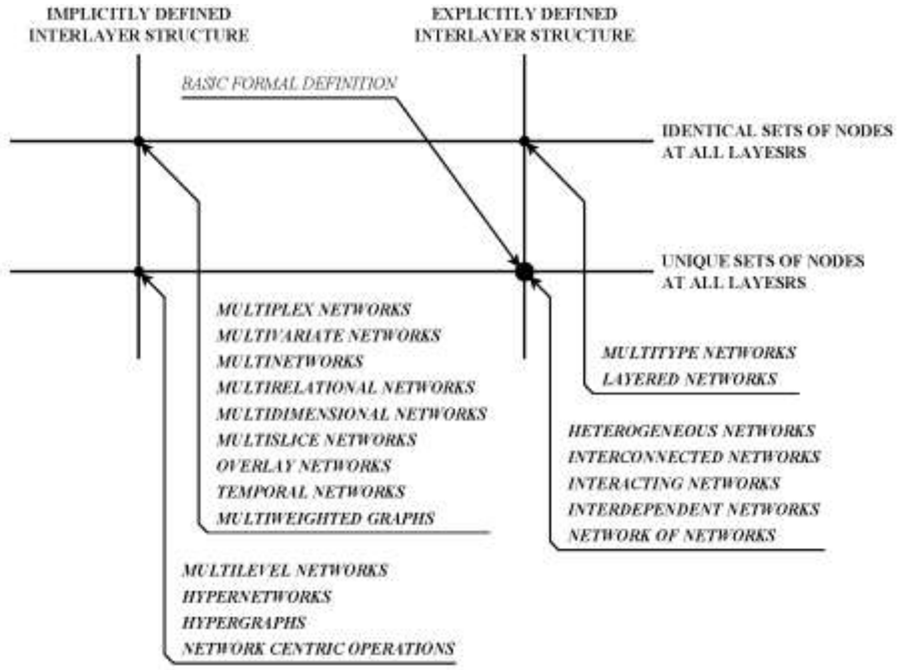

Fig. 5The grid of structural properties.

–*The second intersection point*is denoted by combination of: (1) identical sets of nodes on all layers; and (2) explicitly defined interlayer structures. In this case, multilayer networks can be represented by the formal basic definition where $V^{\alpha} = V^{\beta} = V(M)$ for each and every $1 \leq \alpha \neq \beta \leq N$.This case covers(see Fig. 5):

- Multiple networks [20] [21];
- Layered networks [22] [23].

–*The next intersection point*is denoted by combination of: (1) identical sets of nodes on all layers; and (2) implicitly defined interlayer structures. In this case, multilayer networks can be defined as a set of intra-layer graphs, i.e.:

$$M = \bigcup_{\alpha=1}^{N} G^{\alpha}$$

where$V^{\alpha} = V^{\beta} = V(M)$ for each and every $1 \leq \alpha \neq \beta \leq N$. Inter-layer graphs are represented by biregular $(d = 1)$ balanced bipartite graphs (see Fig. 2), i.e. $E^{\alpha,\beta} = \{(v, v);\, v \in V(M)\}$. As a consequence, they are eliminated from the formal definition.This case covers(see Fig. 5):

- Multiplex networks[8][24];
- Multivariate networks[25][26];
- Multinetworks[27];
- Multirelational networks [12];
- Multidimensional networks[28];
- Multislice networks [29][30];
- Overlay networks [31];
- Temporal networks [30][32];
- Multiweighted graphs [33].

–*The last intersection point* is denoted by combination of: (1) unique sets of nodes on all layers; and (2) implicitly defined interlayer structures. As in the previous case, multilayer networks can be defined as a set of intra-layer graphs where $V^{\alpha} \neq V^{\beta}$ and $V^{\alpha}, V^{\beta} \subseteq V(M)$ for each and every $1 \leq \alpha \neq \beta \leq N$. Inter-layer graphs are represented by biregular $(d = 1)$ balanced bipartite graphs (see Fig. 4), i.e. $E^{\alpha,\beta} = \{(v, v);\, v \in (V^{\alpha} \cap V^{\beta})\}$. As a consequence, they are eliminated from the formal definition.This case covers(see Fig. 5):

- Multilevel networks [34];
- Hypernetworks[34][35];
- Hypergraphs[34][36];
- Network centric operation[37].

It is important to note that this grid covers the majority of multilayer structures presented in the surveys [9][2] (see Fig. 5).

V. Conclusions

One of the major goals of multilayer networks is providing proper and suitable representations of complex systems with many interdependent components, which, in turn, might interact through many different channels. At the same time, the terminology referring to systems with multiple different relations has not yet reached a consensus – different papers from various areas represent similar terminologies to refer to different models, or distinct names for the same model.

To fill the gap, this work introduced a simple taxonomy of multilayer networks based on their structural properties which were covered by four dimensions: (1) intralayer definition; (2) intralayer topology; (3) interlayer definition; and (4) interlayer topology.

Next, the appropriate formal definition was chosen based on the notions of multilayer structures (intra-layer and inter-layer graphs). Four possible options cover the majority of types of multilayer networks have been presented in recent articles.In turn, this definition is completely compatible with the proposed taxonomy.

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