Bounds for the Zeros of a Lacunary Polynomial

M.H. Gulzar [1] , Ajaz Wani [2]

Department of Mathematics, University of Kashmir, Srinagar

Abstract: In this paper we give a bound for the zeros of a lacunary polynomial. The result so obtained generalizes many known results on the Cauchy type bounds for the zeros of a polynomial.

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1. Introduction

The following result known as the Cauchy's Theorem [2] (see also [6,page 123]),is well-known on the location of zeros of a polynomial:

Theorem A. All the zeros of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in the circle $|z|<1+M$, where $M=\max_{0\le j\le n-1}\left|\frac{a_j}{a_n}\right|$.

In the literature [5,6,8] , various bounds for all or some of the zeros of a polynomial

$$P(z)=a_0+a_1z+......+a_nz^n$$

are available. In either case the bounds are expressed as the functions of all the coefficients $a_0, a_1,.....,a_n$ of P(z).

An important class of polynomials is that of the lacunary type i.e. of the type

$$P(z)=a_0+a_1z+......+a_pz^p+a_{n_1}z^{n_1}+a_{n_2}z^{n_2}+........+a_{n_k}z^{n_k},$$

where $0<p=n_0<n_1<n_2<......<n_k; a_0a_pa_{n_1}a_{n_2}.......a_{n_k}\neq 0$, the coefficients $a_j, 0\le j\le p$, are fixed, $a_{n_j}, j=1,2,......,k$ are arbitrary and the remaining coefficients are zero. Landau[3,4] initiated the study of such polynomials in 1906-7 in connection with his study of the Picard's theorem and proved that every trinomial

$$a_0+a_1z+a_nz^n, a_1a_n\neq 0, n\ge 2$$

has at least one zero in $|z|\le 2\left|\frac{a_0}{a_1}\right|$ and every quadrinomial

$$a_0+a_1z+a_mz^m+a_nz^n, a_1a_ma_n\neq 0, 2\le m<n$$

has at least one zero in $|z|\le \frac{17}{3}\left|\frac{a_0}{a_1}\right|$.

Q.G.Mohammad [7] in 1967 proved the following theorem:

Theorem B. All the zeros of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in the circle

$$|z| \le \max(L_p, L_p^{\frac{1}{n}})$$

where

$$L_p = n^{\frac{1}{q}} \{\sum_{j=0}^{n} \left|\frac{a_j}{a_n}\right|^p\}^{\frac{1}{p}},$$

p>1,q>1with $\frac{1}{p}+\frac{1}{q}=1.$

A. Aziz [1] in 2013 proved the following result:

Theorem C. For every positive number t, all the zeros of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in the circle

$$|z| \le (n+1)^{\frac{1}{q}} \{\sum_{j=0}^{n} \left|\frac{ta_j - a_{j-1}}{a_n t^{n-j}}\right|^p\}^{\frac{1}{p}},$$

where p>1,q>1with $\frac{1}{p}+\frac{1}{q}=1.$

2. Main Results

In this paper we consider the case when the polynomial in Theorem C is a lacunary polynomial and prove

Theorem 1. All the zeros of the polynomial

$$P(z)=a_0+a_1z+......+a_\lambda z^\lambda + a_{n_1} z^n, a_\lambda \ne 0, 0\le \lambda \le n-1$$

of degree n lie in the circle

$$|z| \le \max(L_p, L_p^{\frac{1}{n}})$$

where

$$L_p = (\lambda+2)^{\frac{1}{q}} \{\sum_{j=0}^{\lambda+1} \left|\frac{a_j - a_{j-1}}{a_n}\right|^p\}^{\frac{1}{p}}, a_{\lambda+1}=0=a_{-1},$$

p>1,q>1with $\frac{1}{p}+\frac{1}{q}=1.$

For $\lambda = n-1$ in Theorem 1, we get the following result which reduces to Theorem C with t=1 :

Corollary 1. All the zeros of the polynomial

$$P(z)=a_0+a_1z+......+a_{n-1} z^{n-1} + a_{n_1} z^n$$

of degree n lie in the circle

$$|z| \le \max(L_p, L_p^{\frac{1}{n}})$$

where

$$L_p = (n+1)^{\frac{1}{q}} \{\sum_{j=0}^{n} \left|\frac{a_j - a_{j-1}}{a_n}\right|^p\}^{\frac{1}{p}}, a_{-1} = 0,$$

p>1,q>1with $\frac{1}{p} + \frac{1}{q} = 1.$

3. Proof of Theorem 1

Consider the polynomial

$$F(z) = (1-z)P(z)$$

$$= (1-z)(a_n z^n + a_\lambda z^\lambda + a_{\lambda-1} z^{\lambda-1} + ... + a_1 z + a_0)$$
$$= -a_n z^{n+1} - a_\lambda z^{\lambda+1} + (a_\lambda - a_{\lambda-1}) z^\lambda + (a_{\lambda-1} - a_{\lambda-2}) z^{\lambda-1} + ... + (a_1 - a_0) z + a_0$$

$$= -a_n z^{n+1} + \sum_{j=0}^{\lambda+1} (a_j - a_{j-1}) z^j$$

Therefore

$$|F(z)| \ge |a_n||z|^{n+1} - \sum_{j=0}^{\lambda+1} |a_j - a_{j-1}||z|^j$$

$$= |a_n||z|^{n+1}[1 - \sum_{j=0}^{\lambda+1} \left|\frac{a_j - a_{j-1}}{a_n}\right| . \frac{1}{|z|^{n-j+1}}]$$

$$\ge |a_n||z|^{n+1}[1 - \{(\sum_{j=0}^{\lambda+1} \left|\frac{a_j - a_{j-1}}{a_n}\right|^p)^{\frac{1}{p}} (\sum_{j=0}^{\lambda+1} \frac{1}{|z|^{(n-j+1)q}})^{\frac{1}{q}}\}]$$

by applying Holder's inequality.

Now , if $L_p \ge 1$ then $\max(L_p, L_p^{\frac{1}{n}}) = L_p$. Therefore, for $|z| \ge 1$ so that $|z|^{(n-j+1)q} \ge |z|^q$ i.e. $\frac{1}{|z|^{(n-j+1)q}} \le \frac{1}{|z|^q}$.

Hence ,for $|z| > L_p$,

$$|F(z)| \ge |a_n||z|^{n+1}[1 - \{(\sum_{j=0}^{\lambda+1} \left|\frac{a_j - a_{j-1}}{a_n}\right|^p)^{\frac{1}{p}} (\sum_{j=0}^{\lambda+1} \frac{1}{|z|^q})^{\frac{1}{q}}\}]$$

$$=|a_n||z|^{n+1}[1-\frac{(\lambda+2)^{\frac{1}{q}}}{|z|}(\sum_{j=0}^{\lambda+1}\left|\frac{a_j-a_{j-1}}{a_n}\right|^p)^{\frac{1}{p}}]$$

$$=|a_n||z|^{n+1}[1-\frac{L_p}{|z|}]$$

$$>0.$$

Again , if , if $L_p \le 1$then $\max(L_p, L_p^{\frac{1}{n}}) = L_p^{\frac{1}{n}}$. Therefore, for $|z| \le 1$so that $|z|^{(n-j+1)q} \ge |z|^{nq}$ i.e. $\frac{1}{|z|^{(n-j+1)q}} \le \frac{1}{|z|^{nq}}$. Hence ,for $|z| > L_p$,

$$|F(z)| \ge |a_n||z|^{n+1}[1-\{(\sum_{j=0}^{\lambda+1}\left|\frac{a_j-a_{j-1}}{a_n}\right|^p)^{\frac{1}{p}}(\sum_{j=0}^{\lambda+1}\frac{1}{|z|^{nq}})^{\frac{1}{q}}\}]$$

$$=|a_n||z|^{n+1}[1-\frac{(\lambda+2)^{\frac{1}{q}}}{|z|^n}(\sum_{j=0}^{\lambda+1}\left|\frac{a_j-a_{j-1}}{a_n}\right|^p)^{\frac{1}{p}}]$$

$$=|a_n||z|^{n+1}[1-\frac{L_p}{|z|^n}]$$

$$>0.$$

From the above development it follows that F(z) does not vanish for

$$|z| > \max(L_p, L_p^{\frac{1}{n}}).$$

Consequently all the zeros of F(z) and hence P(z) lie in

$$|z| \le \max(L_p, L_p^{\frac{1}{n}}).$$

That completes the proof of Theorem 1.

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